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Holomorphicity and the Walczak formula on Sasakian manifolds

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Abstract

The Walczak formula is a very nice tool for understanding the geometry of a Riemannian manifold equipped with two orthogonal complementary distributions. M. Svensson [Holomorphic foliations, harmonic morphisms and the Walczak formula, J. London Math. Soc. (2) 68 (3) (2003) 781–794] has shown that this formula simplifies to a Bochner-type formula when we are dealing with Kähler manifolds and holomorphic (integrable) distributions. We show in this paper that such results have a counterpart in Sasakian geometry. To this end, we build on a theory of (contact) holomorphicity on almost contact metric manifolds. Some other applications for (pseudo-)harmonic morphisms on Sasaki manifolds are outlined.

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1. Introduction

Throughout this paper M, N etc. will be connected, $\mathcal{C}^\infty$ manifolds. All geometric objects considered will also be smooth.

The analogue of an almost Hermitian structure on odd-dimensional spaces is the *almost contact metric structure*. We recall the necessary definitions, cf. [3]:

Definition 1.1. An almost contact structure on a $(2n+1)$-dimensional manifold M is a triple (ϕ, ξ, η) where ϕ is a $(1, 1)$ tensor field, ξ is a vector field and η is a 1-form satisfying the following relations:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

A manifold M together with an almost contact structure is called an *almost contact manifold*. ξ is called the *characteristic* vector field.

An *almost contact metric structure* (ϕ, ξ, η, g) is an almost contact structure together with a compatible metric (which always exists), that is a metric g satisfying:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

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If, in addition, η is a *contact form* (i.e. $\eta \wedge (d\eta)^n \neq 0$) and g is an *associated metric* (i.e. $d\eta(X, Y) = g(X, \phi Y)$), then our structure is a *contact metric structure*. In this case ξ coincides with the *Reeb field* of the *contact form* η.

A contact metric structure whose (1,1)-tensor ϕ is normal:

$$[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0 \tag{1.1}$$

is called *Sasakian*.

Sasakian structures are the analogue of Kähler structures on odd-dimensional manifolds. The Sasakian condition is equivalent to the integrability of the corresponding almost complex structure on the Riemannian cone over M, cf. e.g. [4].

The normality equation (1.1) is equivalent to the following one:

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.2}$$

which makes the analogy with the Kähler case transparent: indeed, it is enough to take in both members of (1.2) the component tangent to the *contact distribution* $\mathcal{D} = \operatorname{Ker}\eta$, for $X, Y \in \Gamma(\mathcal{D})$, and then we obtain a parallelism-type condition for ϕ. This is in fact the *transversally Kähler* condition.

An almost contact structure has a natural *transversal holomorphic structure*, transversality being here understood with respect to the foliation defined by the characteristic field. In the language of G-structure, this is an $H^{1,n}$-structure, cf. [16].

The paper is organized as follows. In Section 2 we study invariant (under the action of ϕ) distributions on almost contact manifolds. In Section 3 we study the notion of a holomorphic distribution (in particular, a holomorphic vector field), which is automatically ϕ-invariant. We show how this notion is related to holomorphicity on the cone. Section 4 is devoted to holomorphicity on normal almost contact manifolds, especially on Sasakian manifolds. Finally, in Section 5 we apply our theory of holomorphicity to derive results in Riemannian geometry: applications of the Walczak formula and properties of some particular harmonic morphisms.

2. Invariant distributions on almost contact metric manifolds

In analogy with the definition of a complex distribution on an almost Hermitian manifold we give:

Definition 2.1. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. A distribution $\mathcal{V}$ on M is called *invariant* if $\phi(\mathcal{V}) \subseteq \mathcal{V}$.

Remark 2.1. 1. $\mathcal{D} := \operatorname{Ker}\eta$ is an invariant distribution.

2. On an almost contact metric manifold, a distribution $\mathcal{V}$ is invariant if and only if its orthogonal complementary distribution $\mathcal{H}$ is also invariant.

The proof follows from the anti-symmetry of ϕ. Let $X \in \Gamma(\mathcal{H})$, $V \in \Gamma(\mathcal{V})$; we have:

$$\begin{aligned} g(\phi X, V) &= g(\phi^2 X, \phi V) + \eta(\phi X)\eta(V) = g(-X + \eta(X)\xi, \phi V) \\ &= -g(X, \phi V) + \eta(X)\eta(\phi V) = -g(X, \phi V). \end{aligned}$$

By hypothesis, $\phi V \in \Gamma(\mathcal{V})$, so the last term is zero, which implies that ϕX is orthogonal to V, for every $V \in \Gamma(\mathcal{V})$. This means $\phi X \in \Gamma(\mathcal{H})$.

Note that, unlike in the Hermitian case, an invariant distribution can be even or odd dimension as well. In particular, the dimensions of two complementary invariant distributions on M^{2n+1} cannot have the same parity.

The position of the characteristic field ξ with respect to an invariant distribution is subject to some restrictions:

Lemma 2.1. *On an almost contact metric manifold with an invariant distribution $\mathcal{V}$, the vector field ξ must be in $\Gamma(\mathcal{V})$ or in $\Gamma(\mathcal{H})$, where $\mathcal{H} = \mathcal{V}^{\perp}$.*

Moreover, if $\xi \in \Gamma(\mathcal{V})$, then $\mathcal{H} \subseteq \mathcal{D}$.

Proof. Let $\xi^{\mathcal{H}}$, $\xi^{\mathcal{V}}$ denote respectively the $\mathcal{H}$ and $\mathcal{V}$ components of ξ (the exponent $\mathcal{V}$ or $\mathcal{H}$ will always indicate the orthogonal projections onto these distributions). Then $0 = \phi\xi$ together with the invariance of $\mathcal{H}$ and $\mathcal{V}$ implies

$\phi\xi^{\mathcal{H}} = 0$, $\phi\xi^{\mathcal{V}} = 0$. But $\operatorname{Ker}\phi$ is one-dimensional and therefore, if $\xi^{\mathcal{H}}$ and $\xi^{\mathcal{V}}$ were both non-zero, they would be collinear, a contradiction.

The second statement follows from $\eta(X) = g(X, \xi) = 0$, for all $X \in \Gamma(\mathcal{H})$. □

On the other hand, the characteristic vector field ξ is tangent to any invariant submanifold of a contact metric manifold (cf. [3, p. 122]), so one expects the same phenomenon to occur for (integrable) invariant distributions. We have indeed:

Proposition 2.1. *On a contact metric manifold M^{2n+1} endowed with an invariant distribution $\mathcal{V}$ any of the following conditions implies $\xi \in \Gamma(\mathcal{V})$:*
(i) $\dim(\mathcal{V}) = 2k + 1,\ k \leq n$;
(ii) *$\mathcal{V}$ is integrable.*

In particular, an integrable invariant distribution must be odd-dimensional.

Proof. (i) By Lemma 2.1, it is enough to prove that ξ is not in $\Gamma(\mathcal{H})$.

If $\xi \in \Gamma(\mathcal{H})$, then $\mathcal{H}$ admits (local) frames of the type $\{\xi, X_i, \phi(X_i)\}$, so it is odd-dimensional, like $\mathcal{V}$, a contradiction.

(ii) Suppose that $\xi \in \Gamma(\mathcal{H})$. Then, from Lemma 2.1, $\mathcal{V} \subseteq \mathcal{D}$, where $\mathcal{D}$ is the contact distribution. So, for any $V, W \in \Gamma(\mathcal{V})$, we have

$$g(V, \phi W) = d\eta(V, W) = \frac{1}{2}\left[V\eta(W) - W\eta(V) - \eta([V, W])\right] = -\frac{1}{2}\eta([V, W]) = 0,$$

the last equality being a consequence of the integrability of $\mathcal{V}$. We conclude that ϕW is orthogonal to $\mathcal{V}$, a contradiction. Hence ξ cannot be in $\Gamma(\mathcal{H})$. As Lemma 2.1 shows also that ξ cannot be a "mixed" sum either, the proof is complete.

(Note that we have not used all the contact structure information, but only that g is an associated metric.) □

Example 2.1. On $\mathbb{R}^{2n+1}$ with the standard contact metric structure, the distribution $\mathcal{V}_k$ ($k \leq n$) locally spanned by

$$X_i = 2\frac{\partial}{\partial y^i}, \quad X_{n+i} = 2\left(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}\right) \quad \text{and possibly } \xi \quad (i = \overline{1, k})$$

is an invariant distribution of dimension $2k$, or $2k + 1$ if it contains ξ.

For further use we next prove a relation between the Lie derivative and the covariant derivative of the tensor ϕ, similar to the relation (3.1) in [14]. The following relation is easily derived:

$$g(\nabla_{\phi Z}X, V) = g(X, (\mathcal{L}_V\phi - \nabla_V\phi)Z) - g(X, \phi\nabla_Z V).$$

Using here the anti-symmetry of ϕ, the fact that ∇ is a metric connection and also $g(\phi X, V) = 0$ (because $\mathcal{H}$ is an invariant distribution), we prove:

Proposition 2.2. *Let (M, ϕ, ξ, η, g) be an almost contact metric manifold and $\mathcal{V}$ an invariant distribution with orthogonal complement $\mathcal{H}$. For any section X of $\mathcal{H}$ and any vector field V tangent to $\mathcal{V}$, we have:*

$$g(\nabla_{\phi Z}X + \nabla_Z\phi X, V) = g(X, (\mathcal{L}_V\phi - \nabla_V\phi)Z), \quad \forall Z \in \Gamma(TM). \tag{2.1}$$

We recall here that *the second fundamental form $B^{\mathcal{V}}$* and *the integrability tensor $I^{\mathcal{V}}$*, of $\mathcal{V}$, are defined by:

$$B^{\mathcal{V}}(V, W) = \frac{1}{2}\left(\nabla_V W + \nabla_W V\right)^{\mathcal{H}}, \quad I^{\mathcal{V}}(V, W) = [V, W]^{\mathcal{H}}, \quad V, W \in \Gamma(\mathcal{V}).$$

As for the distribution $\mathcal{D}$, which is invariant, we have:

Remark 2.2. On a contact metric manifold,

$$B^{\mathcal{D}}(X, \phi Y) = B^{\mathcal{D}}(\phi X, Y), \quad \forall X, Y \in \Gamma(\mathcal{D}).$$

In particular, $\mathcal{D}$ is a minimal distribution. If, in addition, the manifold is K-contact, then $\mathcal{D}$ is a totally geodesic distribution.

Proof. A result of Olszak [13] states that on a contact metric manifold, we have:

$$(\nabla_X \phi)\, Y + \left(\nabla_{\phi X} \phi\right) \phi Y = 2g(X,Y)\xi - \eta(Y)(X + hX + \eta(X)\xi). \tag{2.2}$$

In particular, if $X, Y \in \Gamma(\mathcal{D})$, the above relation becomes:

$$\nabla_X \phi Y - \phi \nabla_X Y - \nabla_{\phi X} Y - \phi \nabla_{\phi X} \phi Y = 2g(X,Y)\xi.$$

Interchanging X and Y, we obtain a similar relation which, subtracted from the one above, gives:

$$\nabla_X \phi Y + \nabla_{\phi Y} X - (\nabla_{\phi X} Y + \nabla_Y \phi X) = \phi([X,Y] + [\phi X, \phi Y]).$$

Taking only the component collinear with ξ, we get the stated relation for the second fundamental form of $\mathcal{D}$. This implies also $B^{\mathcal{D}}(\phi X, \phi Y) = -B^{\mathcal{D}}(X,Y)$ that assures trace $B^{\mathcal{D}} = 0$ (i.e. $\mathcal{D}$ is minimal).

If, in addition, the manifold is K-contact, ξ is Killing, so the induced foliation $\mathcal{F}_\xi$ is Riemannian, which is equivalent to the fact that the orthogonal distribution $\mathcal{D}$ is totally geodesic. □

The Sasaki condition imposes further restrictions on B:

Proposition 2.3. *Let (M, ϕ, ξ, η, g) be a Sasaki manifold endowed with an invariant distribution $\mathcal{V}$ which contains ξ. Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{V}$. Then the following relations hold:*

$$2\left(B^{\mathcal{V}}(U, \phi V) - \phi B^{\mathcal{V}}(U,V)\right) = \phi(I^{\mathcal{V}}(U,V)) - I^{\mathcal{V}}(U, \phi V), \quad \forall U, V \in \Gamma(\mathcal{V}). \tag{2.3}$$

In particular:

$$2B^{\mathcal{V}}(U,\xi) + I^{\mathcal{V}}(U,\xi) = 0; \quad B^{\mathcal{V}}(\phi U, \xi) = \phi\left(B^{\mathcal{V}}(U,\xi)\right), \quad \forall U \in \Gamma(\mathcal{V}).$$

Proof. Note that $\xi \in \Gamma(\mathcal{V})$ implies $\mathcal{H} \subseteq \mathcal{D}$. The result now follows from the definitions and the Sasaki condition: $\nabla_U \phi V = \phi \nabla_U V + g(U,V)\xi - \eta(V)U$.

For the second assertion, put $V = \xi$ in formula (2.3) and for the last one, take into account the fact that on a Sasaki manifold we have $(\mathcal{L}_\xi \phi)X = 0$. □

If $\mathcal{V}$ is integrable, we recover the formulas for invariant submanifolds stated in [18, p. 49]:

Corollary 2.1. *If N is an invariant submanifold of a Sasaki manifold M, then:*

(i) $B(X,\xi) = 0$,
(ii) $B(X, \phi Y) = B(\phi X, Y) = \phi B(X,Y)$ *for any vector field X tangent to N (here B denotes the second fundamental form of the submanifold).*

3. Infinitesimal holomorphicity on normal almost contact manifolds

3.1. Definitions and first properties

Definition 3.1. Let (M, ϕ, ξ, η) be a normal almost contact manifold. A (local) vector field X on M is called *contact-holomorphic* if

$$(\mathcal{L}_X \phi) Y = \eta\left([X, \phi Y]\right) \xi, \quad \forall Y \in \Gamma(TM). \tag{3.1}$$

A distribution $\mathcal{V}$ on M is called *contact-holomorphic* if it admits, around every point, a local frame consisting of contact-holomorphic vector fields.

When the context does not impose distinctions, we shall simply write *holomorphic* instead of *contact-holomorphic*.

Holomorphicity of X means collinearity of $(\mathcal{L}_X \phi)Y$ with ξ: the particular form of the coefficient of ξ, generally denoted by $\alpha_X(Y)$, results from this collinearity condition.

The next result shows the *ϕ-invariance* of the above defined holomorphicity (unlike the usual property $(\mathcal{L}_X \phi)Y = 0$):

Lemma 3.1. *Let X be a holomorphic vector field on a normal almost contact metric manifold. Then ϕX is also holomorphic. In particular, a holomorphic distribution is necessarily invariant.*

Proof. An explicit formula for the Lie derivative of ϕ with respect to ϕX is provided by the following reformulation of the Eq. (1.1):

$$(\mathcal{L}_{\phi X}\phi)Y = \phi(\mathcal{L}_X\phi)Y - 2d\eta(X, Y)\xi.$$

Hence, if X holomorphic, then the above equation gives us:

$$(\mathcal{L}_{\phi X}\phi)Y = -2d\eta(X, Y)\xi.$$

We now verify that the coefficient of ξ is the same as the one predicted by the definition. Recall that $\alpha_X(Y) = \eta([X, \phi Y])$, so we have to show that:

$$\alpha_{\phi X}(Y) = \eta([\phi X, \phi Y]) = -2d\eta(X, Y).$$

But the normality of ϕ assures that

$$N^{(2)} = 0 \Leftrightarrow \eta([\phi X, Y] + [X, \phi Y]) = \phi X(\eta(Y)) - \phi Y(\eta(X)).$$

In the above relation we replace Y with ϕY and we obtain:

$$\eta([\phi X, \phi Y] - [X, Y] + \eta(Y)[X, \xi] + X(\eta(Y))\xi) = Y(\eta(X)) - \eta(Y)\xi(\eta(X)),$$

which reduces to

$$\eta([\phi X, \phi Y]) + X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = -\eta(Y)(\xi(\eta(X)) - \eta([\xi, X])).$$

Finally we use $N^{(4)} := (\mathcal{L}_\xi\eta)X = 0$ to derive

$$\eta([\phi X, \phi Y]) = -2d\eta(X, Y). \quad \square$$

Remark 3.1. (i) From the above proof we obtain an alternative expression for the collinearity factor α_X:

$$\alpha_X(Y) = -\eta([\phi X, Y]) + \phi X(\eta(Y)) - \phi Y(\eta(X)).$$

(ii) $\alpha_X(\xi) = 0$ for any holomorphic vector field X. This implies that $[X, \xi]$ must be collinear with ξ (or, equivalently, $[X^{\mathcal{D}}, \xi] = 0$). In other words, X is projectable with respect to the foliation $\mathcal{F}_\xi$ locally generated by ξ.

(iii) $\alpha_\xi(Y) = 0$ for any vector field Y. Indeed, the normality of ϕ implies $N^{(3)} := (\mathcal{L}_\xi\phi)Y = 0$, so that ξ is holomorphic.

(iv) X is holomorphic if and only if $[X, \xi]$ is collinear with ξ and $((\mathcal{L}_X\phi)Y)^{\mathcal{D}} = 0$, $\forall Y \in \Gamma(\mathcal{D})$. If M is Sasakian, these properties define the *transversally holomorphic fields*, introduced by Nishikawa and Tondeur in [12], for manifolds endowed with a Kähler foliation.

Proposition 3.1. *On a normal almost contact manifold, the set $\mathfrak{hol}(M)$ of holomorphic vector fields is a Lie subalgebra of $\Gamma(TM)$.*

Proof. Let X and X' be holomorphic vector fields. Then:

$$\begin{aligned}\left(\mathcal{L}_{[X,X']}\phi\right)Y &= ([\mathcal{L}_X, \mathcal{L}_{X'}]\phi)Y = \mathcal{L}_X(\mathcal{L}_{X'}\phi)Y - \mathcal{L}_{X'}(\mathcal{L}_X\phi)Y \\ &= [X, (\mathcal{L}_{X'}\phi)Y] - (\mathcal{L}_{X'}\phi)([X, Y]) - \left[X', (\mathcal{L}_X\phi)Y\right] + (\mathcal{L}_X\phi)([X', Y]).\end{aligned}$$

Using the fact that X and X' are holomorphic and the remark that $[X, \xi]$ must be collinear with ξ in this case, we easily obtain that the projection on $\mathcal{D}$ of the above expression is zero. Hence $[X, X']$ is also holomorphic. $\square$

On closed Sasakian manifolds with constant transversal scalar curvature, the structure of $\mathfrak{hol}(M)$ is established in analogy with the Kähler case, cf. [12].

Example 3.1. On $\mathbb{R}^{2n+1}$ with the standard contact metric structure, take an arbitrary vector field written in an adapted frame as

$$X = \alpha \frac{\partial}{\partial z} + \beta^i \left(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z} \right) + \gamma^i \frac{\partial}{\partial y^i},$$

where summation is taken with $i = \overline{1, n}$. Note that β^i and γ^i are the coefficients of $\frac{\partial}{\partial x^i}$ and of $\frac{\partial}{\partial y^i}$ respectively. Then X is holomorphic if and only if, for any $i = \overline{1, n}$, β^i and γ^i satisfy the Cauchy–Riemann equations in the variables x^j, y^j and are constant in z:

$$\frac{\partial \beta^i}{\partial x^j} = \frac{\partial \gamma^i}{\partial y^j}, \quad \frac{\partial \beta^i}{\partial y^j} = -\frac{\partial \gamma^i}{\partial x^j}, \quad j = \overline{1, n}, \quad \frac{\partial \beta^i}{\partial z} = \frac{\partial \gamma^i}{\partial z} = 0.$$

The Corollary 3.3 below shows that the above description of holomorphic vector fields is not an exceptional one.

As in the complex case (see [11], p. 30) we can express the contact-holomorphicity by the vanishing of some $\bar{\partial}$-operator. In this case $\bar{\partial} : \Gamma(TM) \longrightarrow \mathrm{End}(TM)$ satisfies the Leibniz rule and is expressed as follows with respect to Levi-Civita connection:

$$\bar{\partial} X(Y) = \frac{1}{2} \phi \left(\nabla_Y X + \phi \nabla_{\phi Y} X - \phi(\nabla_X \phi) Y \right).$$

One can verify that a vector field X is contact-holomorphic if and only if $\bar{\partial} X(Y) = 0$, for all Y. Equivalently, this means the projectability of X and the vanishing on $X^{\mathcal{D}}$ of a standard (transversal) $\bar{\partial}$-operator appropriate to $\mathcal{D}$ as $T^{\perp}\mathcal{F}_{\xi}$. Explicitly: $\bar{\partial}^{\mathcal{D}} X(Y) = \frac{1}{2} \left(\nabla^{\mathcal{D}}_Y X + \phi \nabla^{\mathcal{D}}_{\phi Y} X - \phi(\nabla^{\mathcal{D}}_X \phi) Y \right)$, for all $Y \in \Gamma(\mathcal{D})$, where $\nabla^{\mathcal{D}}$ is the adapted connection in $\mathcal{D}$ in the sense of Tondeur [17]. Therefore we are dealing with a transversal, projectable notion of holomorphicity for vector fields on M regarded as a foliated manifold (with the foliation $\mathcal{F}_{\xi}$).

In the Sasaki case, the above formula reduces to:

$$\bar{\partial} X(Y) = \frac{1}{2} \phi \left(\nabla_Y X + \phi \nabla_{\phi Y} X \right), \quad \text{for } Y \in \Gamma(\mathcal{D}) \text{ and } \bar{\partial} X(\xi) = \phi([\xi, X]).$$

3.2. *The holomorphicity condition seen on the cone*

Recall that the *cone* $\mathcal{C}(M)$ over an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is $M^{2n+1} \times \mathbb{R}$ with an almost complex structure defined by:

$$J \left(X, f \frac{\mathrm{d}}{\mathrm{d}t} \right) = \left(\phi X - f \xi, \eta(X) \frac{\mathrm{d}}{\mathrm{d}t} \right).$$

We point out that the above formula fits the well-known construction of an almost contact structure on orientable hypersurfaces of almost complex manifolds (if we take the standard immersion of M into the cone $\mathcal{C}(M)$ at $t = 1$). For details, see [3, Example 4.5.2].

Proposition 3.2. *Let (M, ϕ, ξ, η, g) be a normal almost contact metric manifold. As a vector field on the cone over M, $(X, f \frac{\mathrm{d}}{\mathrm{d}t})$ is holomorphic if and only if, for any $Y \in \Gamma(TM)$, the following relations are satisfied:*

(i) $(\mathcal{L}_X \phi) Y = Y(f) \xi$;
(ii) $X(\eta(Y)) - \eta([X, Y]) - \phi Y(f) - \eta(Y) \frac{\mathrm{d}f}{\mathrm{d}t} = 0$;
(iii) $[X, \xi] + \frac{\mathrm{d}f}{\mathrm{d}t} \xi = 0$;
(iv) $\xi(f) = 0$.

Hence, if $(X, f \frac{\mathrm{d}}{\mathrm{d}t})$ is holomorphic on the cone, then X is a contact-holomorphic vector field on M. Moreover, we have the following implications:

$$\text{“(i)} \wedge \text{(iii)} \Rightarrow \text{(ii)”} \quad \textit{and} \quad \text{“(i)} \Rightarrow \text{(iv)”}.$$

Proof. One can derive by straightforward computations the following formulas:

$$\left(\mathcal{L}_{(X, f\frac{\mathrm{d}}{\mathrm{d}t})} J\right)(Y, 0) = \left((\mathcal{L}_X\phi)Y - Y(f)\xi, \left(X\eta(Y) - \eta([X, Y]) - \phi Y(f) - \eta(Y)\frac{\mathrm{d}f}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

$$\left(\mathcal{L}_{(X, f\frac{\mathrm{d}}{\mathrm{d}t})} J\right)\left(0, \frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(-[X, \xi] - \frac{\mathrm{d}f}{\mathrm{d}t}\xi, \xi(f)\frac{\mathrm{d}}{\mathrm{d}t}\right).$$

As the holomorphicity of $(X, f\frac{\mathrm{d}}{\mathrm{d}t})$ is equivalent to the vanishing of both expressions above, the result follows.

Let us prove the second assertion.

The implication "(i) $\wedge$ (iii) $\Rightarrow$ (ii)" is derived by applying (i) to ϕY instead of Y. We obtain $\alpha_X(\phi Y) = \phi Y(f) = X\eta(Y) - \eta([X, Y]) - \eta(Y)\eta([\xi, X])$. But from (iii) we have $\eta([\xi, X]) = \frac{\mathrm{d}f}{\mathrm{d}t}$, so the relation (ii) follows.

In order to get "(i) $\Rightarrow$ (iv)", put $Y = \xi$ in (i): $(\mathcal{L}_X\phi)\xi = \xi(f)\xi$. But, as X is holomorphic on M, we have already noticed that $(\mathcal{L}_X\phi)\xi = 0$ (i.e. $\alpha_X(\xi) = 0$), so our implication follows. □

Corollary 3.1. *The contact-holomorphic vector fields on M, which come by projection of the holomorphic fields on $\mathcal{C}(M)$, form a Lie subalgebra of $\mathfrak{hol}(M)$, denoted by $\mathfrak{hol}_{pr}(M)$. They are contact-holomorphic fields X with two additional properties:*

(a) *The 1-form α_X is exact: there exists a function f on M such that*

$$Y(f) = \eta([X, \phi Y]), \quad \forall Y \in \Gamma(TM).$$

(b) $\eta([X, \xi])$ *is (locally) constant (i.e. the factor of collinearity between $[X, \xi]$ and ξ is constant).*

Proof. We have seen that, in order to be holomorphic on the cone, a vector field must satisfy only (i) and (iii). From condition (i) we obtain (a). From (iii), it follows that $\frac{\mathrm{d}f}{\mathrm{d}t} = \eta([\xi, X])$, so f is a linear function in t: $f(p, t) = \eta([\xi, X])t + F(p)$, $p \in M$. In order to verify the equation in (a), such a function must have the coefficient $\eta([X, \xi])$ (locally) constant, that is (b) holds.

Conversely, if X is a contact-holomorphic vector field on M, which satisfies in addition (a) and (b), then $\left(X, (\eta([\xi, X])t + f)\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is holomorphic on $\mathcal{C}(M)$.

In order to see that $\mathfrak{hol}_{pr}(M)$ is a Lie subalgebra, it is enough to note that, on the cone, the holomorphic vector fields form a Lie algebra and that the following relation holds:

$$\left[\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right), \left(X', g\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] = \left([X, X'], \left(X(g) - X'(f) + f\frac{\mathrm{d}g}{\mathrm{d}t} - g\frac{\mathrm{d}f}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\right). \quad \square$$

Remark 3.2. The subalgebra, $\mathfrak{hol}_{pr}(M)$ contains all vector fields along which ϕ is invariant: $\mathcal{L}_X\phi = 0$.

The nature of the constraints (a) and (b) becomes very clear when expressed in local coordinates for the case of $\mathbb{R}^{2n+1}$:

Example 3.2. On $\mathbb{R}^{2n+1}$ with the standard contact metric structure, let $X = \alpha\frac{\partial}{\partial z} + \beta^i\frac{\partial}{\partial x^i} + \gamma^i\frac{\partial}{\partial y^i}$ be a holomorphic vector field.

Then $X \in \mathfrak{hol}_{pr}(\mathbb{R}^{2n+1})$ if and only if, in addition, the coefficient α takes the form: $\alpha = Cz + H(x_i, y_i)$, where H is a harmonic function and $C \in \mathbb{R}$.

Remark 3.3. The relation between contact-holomorphicity on the Sasaki manifolds and holomorphicity on its Kähler cone can also be obtained using the relations between the Levi-Civita connections on M and $\mathcal{C}(M)$, and ∇ $\bar{\nabla}$, respectively (for the details, see [4]). Identifying X on M with $(X, 0)$ on the cone, one can prove the following formula:

$$(\mathcal{L}_X J)Y = (\mathcal{L}_X\phi)Y - [X(r\eta(Y)) + r\eta([X, Y])]\,\partial_r. \tag{3.2}$$

Indeed, we have the following sequence of identities (where $\Psi := r\partial_r$ is the Euler field on the cone):

$$\begin{aligned}(\mathcal{L}_X J)Y &= \bar{\nabla}_X JY - J\bar{\nabla}_X Y - \bar{\nabla}_{JY}X + J\bar{\nabla}_Y X\\ &= \bar{\nabla}_X(\phi Y - \eta(Y)\Psi) - J(\nabla_X Y - rg(X, Y)\partial_r) - \bar{\nabla}_{\phi Y - \eta(Y)\Psi}X + J(\nabla_Y X - rg(Y, X)\partial_r)\end{aligned}$$

$$= \bar{\nabla}_X \phi Y - X(\eta(Y))\Psi - \eta(Y)\bar{\nabla}_X \Psi - J\nabla_X Y - \bar{\nabla}_{\phi Y} X + \bar{\nabla}_{\eta(Y)\Psi} X + J\nabla_Y X$$
$$= \nabla_X \phi Y - rg(X, \phi Y)\partial_r - X(\eta(Y))\Psi - \eta(Y)\left[X(r)\partial_r + r\frac{1}{r}X\right]$$
$$- \phi(\nabla_X Y) + \eta(\nabla_X Y)\Psi - \nabla_{\phi Y} X + rg(X, \phi Y)\partial_r + \eta(Y)X + \phi(\nabla_Y X) - \eta(\nabla_Y X)\Psi.$$

This in turn implies formula (3.2).

Corollary 3.2. *On a normal almost contact metric manifold (M, ϕ, ξ, η, g) we have:*

(i) *$a\xi$ is a contact-holomorphic vector field, for any function a defined on M (so $a\xi \in \mathfrak{hol}(M)$, but it is not necessarily the case that $a\xi \in \mathfrak{hol}_{pr}(M)$);*
(ii) *$(\xi, c\frac{\mathrm{d}}{\mathrm{d}t})$ is a holomorphic vector field on the cone if and only if c is a constant.*

Proof. (i) A consequence of normality of ϕ (see [3]) is that $(\mathcal{L}_\xi \phi)Y = 0$. Now, it is an easy task to compute $(\mathcal{L}_{a\xi}\phi)Y = a(\mathcal{L}_\xi \phi)Y - \phi Y(a)\xi$ and to notice that $\alpha_{a\xi}(Y) = -\phi Y(a)\xi$, so the assertion is proved.

(ii) The argument is obvious. □

3.3. Holomorphicity on Sasakian manifolds

Recall that on a Riemannian manifold, an arbitrary vector field V induces a derivation A_V (a tensor field of type $(1, 1)$), defined by: $A_V(X) := \nabla_X V$. In the complex case, A_V is J-linear if and only if V is holomorphic. In our case something similar is happening:

Proposition 3.3. *On a Sasaki manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ we have:*

(i) *V is holomorphic if and only if*

$$(A_V \circ \phi - \phi \circ A_V)(X) \quad \text{is collinear with } \xi, \quad \text{for all } X \in \Gamma(\mathcal{D})$$

and also if: $V^{\mathcal{D}} = \phi\nabla_\xi V$ (which is equivalent to: $[X, \xi]$ collinear with ξ).

(ii) *If M^{2n+1} is compact and regular and X is a contact-holomorphic vector field on M^{2n+1}, then $\pi_* X$ is holomorphic, where $\pi : M^{2n+1} \longrightarrow M^{2n}$ represents the Boothby–Wang fibration. Conversely, the horizontal lift of any holomorphic vector field on M^{2n} is a contact-holomorphic vector field on M^{2n+1}.*

In particular, the contact distribution on such a Sasaki manifold is holomorphic.

Proof. (i) Using the Sasaki condition (1.2) and assuming (3.1) (V is holomorphic), we derive:

$$\nabla_{\phi X} V = \phi\nabla_X V - \eta(X)V + (g(V, X) - \eta([V, \phi X]))\,\xi.$$

From this, the stated collinearity follows immediately.

Conversely, we can verify that $\eta\left(\nabla_{\phi X} V\right) = g(V, X) - \eta([V, \phi X])$ and thereafter we can conduct the same calculation backwards to obtain the holomorphicity condition (3.1).

(ii) As a direct consequence of the fact that the Boothby–Wang fibration is a Riemannian submersion and satisfies also $\pi_* \phi X = J\pi_* X$, one gets the relation

$$(\mathcal{L}_{\pi_* X} J)\pi_* Y = \pi_*(\mathcal{L}_X \phi)Y,$$

for all projectable vector fields X, Y. Note also that (horizontal) contact-holomorphic vector fields on M^{2n+1} are, by definition, projectable ($[X^{\mathcal{D}}, \xi] = 0$). The result now follows, as ξ spans $\operatorname{Ker} \pi_*$. □

A source of examples of holomorphic vector fields is the following:

Proposition 3.4. *Let (M, ϕ, ξ, η, g) be a contact metric manifold. Then any two of the following conditions imply the third one:*

(i) *$(\mathcal{L}_X g)(Y, Z) = 0, \ \forall Y, Z \in \Gamma(\mathcal{D})$,*
(ii) *$i_X d\eta$ is a closed form,*
(iii) *X is a holomorphic vector field.*

Moreover, a vector field X on M is a Killing vector field, which commutes with ξ if and only if X is a holomorphic vector field which is also a strict infinitesimal contact transformation (i.e. $\mathcal{L}_X\eta = 0$).

Proof. The first assertion is a consequence of the following relation:

$$(\mathcal{L}_X g)\,(Y, \phi Z) = (\mathcal{L}_X d\eta)\,(Y, Z) - g(Y, (\mathcal{L}_X\phi)Z).$$

For the second assertion we apply the results obtained by Tanno in [16, Th. 3.1 and Prop. 3.6], because the holomorphic vector fields which are also strict infinitesimal contact transformations are precisely those for which $\mathcal{L}_X\phi = 0$. □

Remark 3.4. The first assertion in the above proposition can be reformulated in the following terms, adequate to the foliated structure of the contact metric manifold M:

a contact-holomorphic vector field with zero transversal divergence is a transversal Killing vector field.

Clearly, this is a similar result to the "if" part of the Bochner–Yano theorem in the Kähler case, cited in [9, p. 93]. The converse is also true on closed Sasakian manifolds, cf. [12].

We recall (in Tondeur's notation, see [17]) that transversal divergence of an infinitesimal automorphism of a foliation is defined by the relation $\Theta(X)vol = \mathrm{div}_B\, X \cdot vol$, where vol is a holonomy invariant transversal volume ($vol = d\eta^n$, in our case).

The following analogy with the complex case will be very helpful for local considerations:

Proposition 3.5. *On a normal almost contact metric manifold M^{2n+1} there always exist (local) adapted frames consisting of contact-holomorphic vector fields.*

Proof. Note first that on the cone over M the vector fields $(\xi, 0)$ and $\left(0, \frac{\mathrm{d}}{\mathrm{d}t}\right)$ are (real) holomorphic. Moreover, by construction, $\left(\mathrm{i}\xi, \frac{\mathrm{d}}{\mathrm{d}t}\right) \in T^{\mathbb{C}}\mathcal{C}(M)$ is a holomorphic vector field on the complexified tangent space of the cone.

But in our hypothesis, $\mathcal{C}(M)$ is a complex manifold so its tangent bundle is holomorphic and then admits local frames of complex holomorphic sections. We can always complete $\left(\mathrm{i}\xi, \frac{\mathrm{d}}{\mathrm{d}t}\right)$ to such a frame.

Let $\left\{\left(X_j, f_j\frac{\mathrm{d}}{\mathrm{d}t}\right) - \mathrm{i}J\left(X_j, f_j\frac{\mathrm{d}}{\mathrm{d}t}\right) \mid j = \overline{1, n}\right\}$ be such a local completion.

We want to prove that $\{\xi, X_j^{\mathcal{D}}, \phi X_j^{\mathcal{D}} \mid j = \overline{1, n}\}$ is an independent family, so it represents a local adapted frame for M, consisting of contact-holomorphic vector fields. Observe that $\left(X_j, f_j\frac{\mathrm{d}}{\mathrm{d}t}\right) - \mathrm{i}J\left(X_j, f_j\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(X_j - \mathrm{i}\phi X_j + \mathrm{i}f_j\xi, (f_j - \mathrm{i}\eta(X_j))\frac{\mathrm{d}}{\mathrm{d}t}\right)$.

Let us now verify that $\left\{X_j^{\mathcal{D}} - \mathrm{i}\phi X_j^{\mathcal{D}} \mid j = \overline{1, n}\right\}$ forms an independent family over $\mathbb{C}$, consisting of complex holomorphic fields. (In the following, the Einstein convention will be used.) Suppose $\lambda^j(X_j^{\mathcal{D}} - \mathrm{i}\phi X_j^{\mathcal{D}}) = 0$. Then we have successively:

$$\lambda^j\left(X_j^{\mathcal{D}} - \mathrm{i}\phi X_j^{\mathcal{D}}, 0\right) = 0,$$
$$\lambda^j\left(X_j - \mathrm{i}\phi X_j - \eta(X_j)\xi, 0\right) = 0,$$
$$\lambda^j\left(X_j - \mathrm{i}\phi X_j + \mathrm{i}f_j\xi, 0\right) - \lambda^j((\mathrm{i}f_j + \eta(X_j))\xi, 0) = 0,$$
$$\lambda^j\left(X_j - \mathrm{i}\phi X_j + \mathrm{i}f_j\xi, (f_j - \mathrm{i}\eta(X_j))\frac{\mathrm{d}}{\mathrm{d}t}\right) - \lambda^j\left((\mathrm{i}f_j + \eta(X_j))\xi, (f_j - \mathrm{i}\eta(X_j))\frac{\mathrm{d}}{\mathrm{d}t}\right) = 0$$

and finally

$$\lambda^j\left[\left(X_j, f_j\frac{\mathrm{d}}{\mathrm{d}t}\right) - \mathrm{i}J\left(X_j, f_j\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] - \lambda^j\left(f_j - \mathrm{i}\eta(X_j)\right)\left(\mathrm{i}\xi, \frac{\mathrm{d}}{\mathrm{d}t}\right) = 0.$$

But this is a linear combination of the vectors that form the complex holomorphic frame on the cone. Therefore, $\lambda^j = 0$ for all $j = \overline{1, n}$.

Now a simple trick will give us the linear independence over $\mathbb{R}$ of the family $\{X_j^{\mathcal{D}}, \phi X_j^{\mathcal{D}} \mid j = \overline{1, n}\}$.

Suppose that $\alpha^j X_j^{\mathcal{D}} + \beta^j \phi X_j^{\mathcal{D}} = 0$. Then $-\beta^j X_j^{\mathcal{D}} + \alpha^j \phi X_j^{\mathcal{D}} = 0$. Together, these relations give $\alpha^j X_j^{\mathcal{D}} + \beta^j \phi X_j^{\mathcal{D}} - \mathrm{i}(-\beta^j X_j^{\mathcal{D}} + \alpha^j \phi X_j^{\mathcal{D}}) = 0$ which is equivalent to $(\alpha^j + \mathrm{i}\beta^j)(X_j^{\mathcal{D}} - \mathrm{i}\phi X_j^{\mathcal{D}}) = 0$, and hence $(\alpha^j + \mathrm{i}\beta^j) = 0 \Rightarrow \alpha^j = \beta^j = 0$, the relation we wanted to prove.

The argument that ξ is transversal to $\mathcal{D}$ completes the proof. □

A direct computation proves:

Corollary 3.3. *On a normal almost contact manifold, let $\{\xi, E_i, \phi E_i\}$ be a (local) adapted frame consisting of contact-holomorphic vector fields. Then a vector field $X = \alpha\xi + \beta^i E_i + \gamma^i \phi E_i$ is holomorphic if and only if, for all $i = \overline{1, n}$, β^i and γ^i satisfy the generalized Cauchy–Riemann equations:*

$$E_j(\beta^i) = \phi E_j(\gamma^i), \quad E_j(\gamma^i) = -\phi E_j(\beta^i), \quad j = \overline{1, n}$$

and are constant along the flow of ξ (i.e. $\xi(\beta^i) = \xi(\gamma^i) = 0$).

3.4. The flow of a contact-holomorphic vector field

Definition 3.2. A map $\psi : (M, \phi, \xi, \eta, g) \longrightarrow (M', \phi', \xi', \eta', g')$ between almost contact manifolds is called *contact-holomorphic* if

$$d\psi \circ \phi(X) - \phi' \circ d\psi(X) \quad \text{is collinear with } \xi', \quad \forall X \in \Gamma(TM).$$

As before, the word *contact* in the above notion will be omitted when no confusion is possible.

Remark 3.5. If ψ is holomorphic, then $d\psi(\xi)$ must be collinear with ξ'. To see this, put $X = \xi$ in the formula of the above definition.

In particular, the contact-holomorphic maps between *normal* almost contact manifolds are *transversally holomorphic* as maps between foliated manifolds with transversally holomorphic foliations, according to [2] (i.e. $\pi_{\mathcal{D}'} \circ d\psi|_{\mathcal{D}}$ is holomorphic in the usual sense, that is $(\pi_{\mathcal{D}'} \circ d\psi|_{\mathcal{D}}) \circ \phi|_{\mathcal{D}} = \phi'|_{\mathcal{D}'} \circ (\pi_{\mathcal{D}'} \circ d\psi|_{\mathcal{D}})$, where $\pi_{\mathcal{D}}$ stands for the orthogonal projection on the corresponding distribution).

Proposition 3.6. *The flow of a contact-holomorphic vector field on a normal almost contact manifold M consists of contact-holomorphic transformations on M.*

Proof. Observe first that the flow of a holomorphic vector field $\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right)$ on $M \times \mathbb{R}$ decomposes as follows: $\Psi_s = (\psi_s, \psi_s^t)$, where ψ_s can be regarded as the flow of X on M and $\psi_s^t : M \times \mathbb{R} \longrightarrow \mathbb{R}$, $s \in I_\epsilon$ satisfies the differential equation: $\frac{\mathrm{d}\psi_s^t}{\mathrm{d}s} = f(\psi_s, \psi_s^t)$. But we know that if $\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is holomorphic on the cone (which is a complex manifold in this case), then its flow Ψ_s must be a holomorphic transformation on the cone. We then have successively:

$$d\Psi_s \circ J\left(Y, h\frac{\mathrm{d}}{\mathrm{d}t}\right) = J \circ d\Psi_s\left(Y, h\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

$$d\Psi_s\left(\phi Y - h\xi, \eta(Y)\frac{\mathrm{d}}{\mathrm{d}t}\right) = J\left(d\psi_s(Y), d\psi_s^t(Y) + h\frac{\partial\psi_s^t}{\partial t}\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

$$\left(d\psi_s(\phi Y - h\xi), d\psi_s^t(\phi Y - h\xi) + \eta(Y)\frac{\partial\psi_s^t}{\partial t}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$
$$= \left(\phi(d\psi_s(Y)) - \left[Y(\psi_s^t) + h\frac{\partial\psi_s^t}{\partial t}\right]\xi, \eta(d\psi_s(Y))\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

$$d\psi_s(\phi Y) - \phi(d\psi_s(Y)) = hd\psi_s(\xi) - \left[Y(\psi_s^t) + h\frac{\partial\psi_s^t}{\partial t}\right]\xi$$

and

$$\phi Y(\psi_s^t) - h\xi(\psi_s^t) + \eta(Y)\frac{\partial\psi_s^t}{\partial t} = \eta(d\psi_s(Y)).$$

But these two relations must hold also for $Y = 0$, that is: $d\psi_s(\xi) = \frac{\partial \psi_s^t}{\partial t}\xi$ and $\xi(\psi_s^t) = 0$. So the above relations reduces to

$$d\psi_s(\phi Y) - \phi(d\psi_s(Y)) = -Y(\psi_s^t)\xi.$$

Taking into account that $\xi(\psi_s^t) = 0$, the last equation implies, for $Y = \xi$, that $d\psi_s(\xi)$ is collinear with ξ.

All in all, for the flow of X we have obtained precisely the condition of being a contact-holomorphic transformation. Moreover we can see what, geometrically, the factor of collinearity with ξ means. □

Remark 3.6. A contact-holomorphic map between Sasakian manifolds is *transversally harmonic* and an absolute minimum for the energy E_T in its foliated homotopy class, according to [2] (see also [10]).

3.5. The G-structures viewpoint

At the end of this section we shall stress the connection between a certain G-structure of almost contact manifolds and the contact-holomorphicity, which we have been discussing until now (for general definitions, see [9]).

The existence of an almost contact (metric) structure on a manifold M^{2n+1} is equivalent to the existence of a $(U(n) \times 1)$-structure which clearly is not integrable (even when ϕ is normal). The normality of ϕ reflects in the integrability of another G-structure of M^{2n+1}, namely the $H^{1,n}$-structure, called also the *transversal holomorphic structure* (for notation and details, see [6]). The infinitesimal automorphisms of the $H^{1,n}$-structure are precisely the contact-holomorphic vector fields that we have dealt with, so far. In a system of (local) distinguished coordinates $(u, z^j, \overline{z}^j)$, these vector fields take the form

$$X = a(u, z^j, \overline{z}^j)\frac{\partial}{\partial u} + b_k(u, z^j, \overline{z}^j)\frac{\partial}{\partial z^k} + \overline{b}_k(u, z^j, \overline{z}^j)\frac{\partial}{\partial \overline{z}^k}, \quad \text{where } \frac{\partial b_k}{\partial \overline{z}^j} = 0 \quad \text{and} \quad \frac{\partial b_k}{\partial u} = 0.$$

If, in addition, M^{2n+1} is contact, passing from these coordinates to Darboux coordinates will not respect the $H^{1,n}$-structure, so the distinguished coordinates and above local expression for X will be not at all suited for the study of strict contact geometric properties of M^{2n+1}.

4. Complex holomorphicity on normal almost contact manifolds

In this section we stress the notion of holomorphic vector field in the complex context. If (M, ϕ, ξ, η, g) is a normal almost contact metric manifold, then the complexified tangent bundle admits a natural split:

$$T^{\mathbb{C}}M = T^0M \oplus T^{(1,0)}M \oplus T^{(0,1)}M,$$

where $T^{(1,0)}M = \{X - \mathrm{i}\phi X \mid X \in \Gamma(\mathcal{D})\}$, $T^{(0,1)}M = \overline{T^{(1,0)}M}$ and $T^0M = Sp_{\mathbb{C}}\{\xi\}$ are the eigenspaces of ϕ corresponding to the eigenvalues i, −i and 0.

Definition 4.1. On an almost contact manifold (M, ϕ, ξ, η), a smooth function $f : M \longrightarrow \mathbb{C}$ will be called *holomorphic* if $df \circ \phi = \mathrm{i} \cdot df$.

Proposition 4.1. *Let $f : M \longrightarrow \mathbb{C}$ be a smooth function on a normal almost contact manifold M. Then the following statements are equivalent:*

(i) *f is holomorphic,*
(ii) *$Z(f) = 0$, for all $Z \in T^0M \oplus T^{(0,1)}M$,*
(iii) *$df \in \Lambda_B^{(1,0)}M$, where $\Lambda_B^{(1,0)}M$ comes from the natural splitting of the complexification of the sheaf of basic 1-forms on M: $\Lambda_B^1 \otimes \mathbb{C} = \Lambda_B^{(1,0)} \oplus \Lambda_B^{(0,1)}$, cf.* [5].

In addition, if $\psi : M \longrightarrow M$ is a holomorphic map, then $f \circ \psi$ is a holomorphic function on M.

Proof. In order to prove "(i) ⇔ (ii)", we have simply to remark that $df(\xi) = 0$ (so $\xi(f) = 0$) and then the rest of the proof will be similar to the complex case:

$$df(\phi X) = \mathrm{i}df(X) \Leftrightarrow \mathrm{i}df(X + \mathrm{i}\phi X) = 0 \Leftrightarrow (X + \mathrm{i}\phi X)(f) = 0, \quad \forall X \in \Gamma(TM).$$

In the proof of "(i) ⇔ (iii)" it suffices to verify that df is a basic 1-form. We have already seen that $df(\xi) = 0$. It remains to compute:

$$(\mathcal{L}_\xi df)(X) = \xi(df(X)) - df([\xi, X]) = \xi(X(f)) - [\xi, X](f) = X(\xi(f)) = 0.$$

For the last assertion, we have to do a simple verification:

$$d(f \circ \psi)(\phi X) = df(d\psi(\phi X)) = df(\phi(d\psi(X)) + a\xi) = df(\phi(d\psi(X))) = \mathrm{i}df(d\psi(X)). \quad \square$$

Definition 4.2. On a normal almost contact metric manifold M, $Z \in T^0M \oplus T^{(1,0)}M$ will be called *complex holomorphic* if $Z(f)$ is holomorphic for any (local) holomorphic function f on M.

Proposition 4.2. *$Z = a\xi + X - \mathrm{i}\phi X \in T^0M \oplus T^{(1,0)}M$ is complex holomorphic if and only if X is holomorphic (in the expression for Z, a is a complex valued function and $X \in \Gamma(\mathcal{D})$).*

Proof. Let $Z = a\xi + X - \mathrm{i}\phi X$ be a complex holomorphic vector field and f a holomorphic function on M. We have seen that $(X + \mathrm{i}\phi X)(f) = 0$, so $Z(f) = (X - \mathrm{i}\phi X)(f) = 2X(f)$ must be a holomorphic function. This means also that: $(Y + \mathrm{i}\phi Y)(X(f)) = 0, \forall Y \in TM$.

From all this we can deduce that: $[Y + \mathrm{i}\phi Y, X](f) = 0$ (for an arbitrary holomorphic function f), which in turn implies: $[Y + \mathrm{i}\phi Y, X] \in T^0M \oplus T^{(0,1)}M$.

But, for any $W = a\xi + Y + \mathrm{i}\phi Y \in T^0M \oplus T^{(0,1)}M$, we have: $\mathrm{Im}(W)^{\mathcal{D}} = \phi(\mathrm{Re}(W)^{\mathcal{D}})$. In our case, $\mathrm{Im}\,([Y + \mathrm{i}\phi Y, X]) = [\phi Y, X]$ and $\mathrm{Re}\,([Y + \mathrm{i}\phi Y, X]) = [Y, X]$. So we must have:

$$[\phi Y, X]^{\mathcal{D}} = \phi([Y, X]^{\mathcal{D}}) \Leftrightarrow ((\mathcal{L}_X\phi)Y)^{\mathcal{D}} = 0$$

and this means that X is holomorphic.

Conversely, let X be a holomorphic vector field and f a holomorphic function. We have to show that $Z(f) = (a\xi + X - \mathrm{i}\phi X)(f)$ is a holomorphic function too. But $Z(f) = (X - \mathrm{i}\phi X)(f) = 2X(f)$, because $\xi(f) = (X + \mathrm{i}\phi X)(f) = 0$, f being holomorphic. According to Proposition 4.1, $X(f)$ is holomorphic if and only if $(b\xi + Y + \mathrm{i}\phi Y)(X(f)) = 0$, for any b complex valued function and $Y \in \Gamma(\mathcal{D})$. In turn, this is equivalent to $[b\xi + Y + \mathrm{i}\phi Y, X](f) = 0$ which is assured by $[b\xi + Y + \mathrm{i}\phi Y, X] \in T^0M \oplus T^{(0,1)}M$ (due to the holomorphicity of X). $\square$

Analogously to the complex case, we have also:

Proposition 4.3. *On a normal almost contact metric manifold, $T^0M \oplus T^{(1,0)}M$ and $T^0M \oplus T^{(0,1)}M$ are integrable subbundles of $T^{\mathbb{C}}M$, invariant along the flow of a holomorphic vector field X.*

Proof. We have to prove that $[a\xi + X - \mathrm{i}\phi X, b\xi + Y - \mathrm{i}\phi Y] \in T^0M \oplus T^{(1,0)}M$.

A well known result of Ianuş [7] tells us that, in this case, $T^{(1,0)}M$ is involutive. So it remains to prove that $[X - \mathrm{i}\phi X, b\xi] \in T^0M \oplus T^{(1,0)}M$.

Taking into account that $\mathcal{L}_\xi\phi = 0$ (i.e. $[\xi, \phi X] = \phi[\xi, X], \forall X$), we have:

$$\begin{aligned}
[X - \mathrm{i}\phi X, b\xi] &= (X - \mathrm{i}\phi X)(b)\xi + b([X, \xi] - \mathrm{i}[\phi X, \xi]) \\
&= (X - \mathrm{i}\phi X)(b)\xi + b([X, \xi] - \mathrm{i}\phi([X, \xi])) \\
&\in T^0M \oplus T^{(1,0)}M.
\end{aligned}$$

As usual ψ_s denotes the flow of X. We have:

$$\begin{aligned}
d\psi_s(a\xi + X - \mathrm{i}\phi X) &= a d\psi_s(\xi) + d\psi_s(X) - \mathrm{i}d\psi_s(\phi X) \\
&= ab\xi + d\psi_s(X) - \mathrm{i}\left(\phi(d\psi_s X) + b'\xi\right) \\
&= (ab - \mathrm{i}b')\xi + d\psi_s(X) - \mathrm{i}\phi(d\psi_s X) \\
&\in T^0M \oplus T^{(1,0)}M. \quad \square
\end{aligned}$$

Remark 4.1. Note that in Proposition 3.5, we have proved that $T^0M \oplus T^{(1,0)}M$ admits, locally, frames of holomorphic sections.

The proof of the following proposition is an easy computation and we shall omit it:

Proposition 4.4. *On a Sasaki manifold we always have:*

(i) $\nabla_{\overline{W}} Z \in T^0 M \oplus T^{(1,0)} M, \forall W, Z \in T^{(1,0)} M$.
(ii) $\nabla_{a\xi} Z \in T^{(1,0)} M, \forall Z \in T^{(1,0)} M$.
(iii) $\nabla_{\overline{W}} a\xi \in T^0 M \oplus T^{(0,1)} M, \forall W \in T^{(1,0)} M$.

In addition, $Z \in T^{(1,0)} M$ is a complex holomorphic field if and only if:

$$\nabla_{\overline{W}} Z \in T^0 M,\ \forall W \in T^{(1,0)} M \quad \textit{and} \quad \nabla_\xi Z = -\mathrm{i} Z.$$

Remark 4.2. The contact (complex) holomorphicity, which we deal with, is more general than the one introduced by Tanaka in [15]. One can verify that a contact complex holomorphic field from $T^{(1,0)} M$ is holomorphic also in Tanaka's sense if, in addition, it preserves the contact distribution, or, equivalently, if ϕ is invariant along its flow (i.e. $\mathcal{L}_X \phi = 0$). This is a rather strong restriction (generally not satisfied in our context).

5. Holomorphic foliations on a Sasaki manifold

Again by analogy with the Kähler case (treated in [14]), in the following we shall stress some properties of the holomorphic distributions. For the sake of completeness we recall the notion of *mixed sectional curvature* of a Riemannian manifold M endowed with two complementary distributions $\mathcal{V}$ and $\mathcal{H}$:

$$s_{\text{mix}} = \sum_{i,\alpha} K^M (e_i \wedge f_\alpha)$$

where $\{e_i\}, \{f_\alpha\}$ are local orthonormal frames for $\mathcal{V}$ and $\mathcal{H}$.

Proposition 5.1. *On a Sasaki manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, an invariant holomorphic distribution $\mathcal{V}$ of dimension $2p+1$ has the following properties (as usual, $\mathcal{H} = \mathcal{V}^\perp$):*

(i) $\mathcal{V}(\nabla_{\phi Z} X + \nabla_Z \phi X) = 0, \forall Z \in \Gamma(TM), X \in \Gamma(\mathcal{H})$.
(ii) $\phi B^{\mathcal{H}}(X, Y) + g(X, Y)\xi = \frac{1}{2} I^{\mathcal{H}}(X, \phi Y), \forall X, Y \in \Gamma(\mathcal{H})$.
(iii) $|B^{\mathcal{H}}|^2 + 2(n-p) = \frac{1}{4} |I^{\mathcal{H}}|^2$.
(iv) trace $B^{\mathcal{V}} = 0$ *($\mathcal{V}$ is a minimal distribution).*

Proof. (i) Because M is Sasakian, we have: $(\nabla_V \phi) Z = g(V, Z)\xi - \eta(Z) V$. So, for any section X of $\mathcal{H}$ and V of $\mathcal{V}$, the following relation holds: $g\,(X, (\nabla_V \phi) Z) = 0$, also because $\xi \in \Gamma(\mathcal{V})$, by Proposition 2.1. Taking this into account, together with the holomorphicity hypothesis, we derive the relation (i) using Proposition 2.2.

(ii) Using (i), we have:

$$\begin{aligned} g\left(\frac{1}{2} I^{\mathcal{H}}(X, \phi Y), V\right) &= g\left(\frac{1}{2}(\nabla_X \phi Y - \nabla_{\phi Y} X), V\right) = g\left(\frac{1}{2}(\nabla_X \phi Y + \nabla_Y \phi X), V\right) \\ &= \frac{1}{2} g(\phi \nabla_X Y + g(X, Y)\xi - \eta(Y) X + \phi \nabla_Y X + g(Y, X)\xi - \eta(X) Y, V) \\ &= \frac{1}{2} g(\phi(\nabla_X Y + \nabla_Y X) + 2g(X, Y)\xi - \eta(Y) X - \eta(X) Y, V) \\ &= g\left(\phi B^{\mathcal{H}}(X, Y), V\right) + g(X, Y) g(\xi, V). \end{aligned}$$

The last equality completes the proof because all the terms in the relation (ii) are sections of $\mathcal{V}$ and $V \in \mathcal{V}$ was arbitrary.

(iii) This formula involving the Hilbert–Schmidt norms of $B^{\mathcal{H}}$ and $I^{\mathcal{H}}$ is a straight consequence of (ii) if we point out that:

$\eta(B^{\mathcal{H}}(X, Y)) = g\left(B^{\mathcal{H}}(X, Y), \xi\right) = -\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) = 0$, because ξ is a Killing vector field in the Sasakian context.

This assures that $\|\phi B^{\mathcal{H}}(X,Y)\| = \|B^{\mathcal{H}}(X,Y)\|$.

In order to compute $|I^{\mathcal{H}}|^2$, it is worth noticing that $\xi \in \Gamma(\mathcal{V})$ implies $\mathcal{H} \subseteq \mathcal{D}$. So, for a local frame of $\mathcal{H}$ of the type $\{e_i, \phi e_i\}$, we shall have: $\phi^2 e_i = -e_i$.

(iv) The relation (2.3) can be rewritten as follows:

$$2\left(B^{\mathcal{V}}(U,\phi V) - \phi B^{\mathcal{V}}(U,V)\right) = -\left[(\mathcal{L}_U\phi)V\right]^{\mathcal{H}}, \quad \forall U, V \in \Gamma(\mathcal{V}).$$

For a (contact-)holomorphic field U, we get: $B^{\mathcal{V}}(U,\phi V) = \phi B^{\mathcal{V}}(U,V)$, which implies immediately $B^{\mathcal{V}}(U,\phi V) = B^{\mathcal{V}}(\phi U, V)$.

Using also that $[U,\xi]$ is collinear with ξ when U is holomorphic (so $I^{\mathcal{V}}(U,\xi) = 0$), again from Proposition 2.3 we obtain:

$$B^{\mathcal{V}}(U,V) + B^{\mathcal{V}}(\phi U, \phi V) = 0, \quad \forall U, V \in \mathfrak{hol}(M).$$

Therefore, in a local frame of holomorphic vector fields, we will have:

$$\operatorname{trace} B^{\mathcal{V}} = \nabla_\xi \xi + \sum_i \mathcal{H}\left[\nabla_{e_i} e_i + \nabla_{\phi e_i}\phi e_i\right] = 0. \quad \square$$

Proposition 5.2. *Under the same hypothesis as above, the Walczak formula (see [19]) simplifies to:*

$$\operatorname{div}^{\mathcal{V}}\operatorname{trace} B^{\mathcal{H}} + 2(n-p) + \frac{1}{4}|I^{\mathcal{V}}|^2 = s_{\text{mix}} + |B^{\mathcal{V}}|^2. \tag{5.1}$$

Proof. Recall that, for an arbitrary Riemannian manifold (M, g) with two orthogonal complementary distributions $\mathcal{V}$ and $\mathcal{H}$, the Walczak formula asserts:

$$\operatorname{div}^{\mathcal{V}}\operatorname{trace} B^{\mathcal{H}} + \operatorname{div}^{\mathcal{H}}\operatorname{trace} B^{\mathcal{V}} + \frac{1}{4}|I^{\mathcal{H}}|^2 + \frac{1}{4}|I^{\mathcal{V}}|^2 = s_{\text{mix}} + |B^{\mathcal{H}}|^2 + |B^{\mathcal{V}}|^2.$$

Now, applying (iii) and (iv) from Proposition 5.1, the result follows. $\square$

Remark 5.1. When $\mathcal{V}$ is integrable, Eq. (5.1) reduces to:

$$\operatorname{div}^{\mathcal{V}}\operatorname{trace} B^{\mathcal{H}} + 2(n-p) = s_{\text{mix}} + |B^{\mathcal{V}}|^2. \tag{5.2}$$

Integrating (5.2) along any compact leaf, we get the following:

Theorem 5.1 (*Bochner-type Result*). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasaki manifold with a $(2p+1)$-dimensional holomorphic foliation such that $s_{\text{mix}} \geq 2(n-p)$. Then $s_{\text{mix}} = 2(n-p)$ along every compact leaf and every compact leaf is a totally geodesic submanifold of M. In particular, if $s_{\text{mix}} > 2(n-p)$, then $\mathcal{V}$ cannot have compact leaves.*

Corollary 5.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a compact Sasaki manifold with the sectional curvature $k \geq 2m$ $(m < n)$. Then every (ϕ, J)-holomorphic submersion from M into any Hermitian manifold N^{2m} has totally geodesic fibers.*

Other results such as Propositions 3.8 and 3.9 in [14], dealing with holomorphic conformal foliations, can also be restated, now in a obvious way, for the Sasakian case.

It is worth noticing that the (ϕ, J)-holomorphic submersions on Sasaki manifolds into a Kähler manifold are in fact a special class of pseudo-harmonic morphisms, with very nice geometric properties, cf. [1].

Proposition 5.3. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a Sasaki manifold. Then every (ϕ, J)-holomorphic submersion ψ, from M onto a Kähler manifold (N^{2n}, J, g_N), is a pseudo-horizontally homothetic (PHH) harmonic morphism.*

In particular, it has minimal fibers and the inverse images of complex submanifolds in N are invariant, and so minimal, submanifolds of M. If in addition $m = n$, then the horizontal distribution (of the submersion ψ), $\mathcal{H}$, coincides with the contact distribution on M (in particular $\mathcal{H}$ cannot be integrable).

Proof. The harmonicity of such submersions has been remarked already in [8]. Then we have to verify the PHWC condition (Pseudo-Horizontal Weak Conformality) and the PHH one.

The first condition simply means that the induced almost complex structure on the horizontal bundle (defined by $J_{\mathcal{H}} = d\psi^{-1} \circ J \circ d\psi$) is compatible with the metric g. That is indeed the case, because $\mathcal{H} \subset \mathcal{D}$ (due to $\xi \in \Gamma(\mathcal{V})$) and $J_{\mathcal{H}}$ coincides with ϕ restricted to $\mathcal{H}$ (due to the (ϕ, J)-holomorphicity of ψ).

The second (PHH) condition means that $J_{\mathcal{H}}$ is parallel in horizontal directions with respect to $\nabla^{\mathcal{H}}$, so it satisfies a *partial Kähler* condition. To see this we have to particularize the formula (1.2) for $X, Y \in \Gamma(\mathcal{H}) \subset \Gamma(\mathcal{D})$ and to take the $\mathcal{H}$-part of both sides of the relation. □

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